

# A note on Generalized Concurrences and Entanglement Detection

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## Abstract

We study *generalized concurrences* as a tool to detect the entanglement of bipartite quantum systems. By considering the case of  $2 \times 4$  states of rank 2, we prove that generalized concurrences do not, in general, give a necessary and sufficient condition of separability. We identify a set of entangled states which are undetected by this method.

## 1 Introduction

Consider a bipartite quantum system consisting of two subsystems  $A$  and  $B$ , with associated Hilbert spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$  of dimensions  $n_A$  and  $n_B$ , respectively. The overall system has Hilbert space  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$  of dimension  $n := n_A n_B$ . The state of the total system is represented by an Hermitian  $n \times n$  matrix  $\rho$ , called the *density matrix*, which is positive semi-definite and has trace equal to one. A density matrix is called *separable* if it can be written as

$$\rho = \sum_j \mu_j |\psi_{Aj}\rangle\langle\psi_{Aj}| \otimes |\psi_{Bj}\rangle\langle\psi_{Bj}|, \quad \mu_j > 0, \quad \sum_j \mu_j = 1, \quad |\psi_{Aj}\rangle(|\psi_{Bj}\rangle) \in \mathcal{H}_A(\mathcal{H}_B). \quad (1)$$

A state that is not separable is called *entangled*. One of the fundamental open questions in quantum information theory is to give criteria to decide whether a density matrix  $\rho$  describing the state of a bipartite quantum system represents an entangled or a separable state.

Define the *partial transposition* of a  $n_A n_B \times n_A n_B$  matrix  $\rho = \sigma \otimes S$  (with  $\sigma$  and  $S$  of dimensions  $n_A \times n_A$  and  $n_B \times n_B$ , respectively) as  $\rho^{T_A} := \sigma^T \otimes S$  and extend the definition to any Hermitian matrix by linearity. A very popular test introduced in [9],[14], based on the partial transposition of  $\rho$ , gives a criterion which is both simple and very powerful. This test is called the *Positive Partial Transposition (PPT)-test*. It says that if  $\rho$  is separable  $\rho^{T_A} \geq 0$ . We shall call a state  $\rho$  with  $\rho^{T_A} \geq 0$  a *PPT-state*. Therefore, every separable state is a PPT-state. The converse has been proved to be true in the  $2 \times 2$  and  $2 \times 3$  cases [9], as well as in the  $2 \times N$  case with rank lower than  $N$  [12]. The latter result has been generalized to  $M \times N$  ( $M < N$ ) and rank lower than  $N$  in [11]. On the other hand, higher dimensional examples have been constructed of bipartite systems whose entanglement is not detected by this test. This motivates the investigation of more tests to detect entanglement in quantum systems.

Generalizing the definition of *concurrence* given by S. Hill and W. Wootters [8], [16] for the  $2 \times 2$  case, A. Uhlmann introduced *generalized concurrences* in [15]. Generalized concurrences are functions of the state  $\rho$ ,  $C_\Theta$ , parametrized by a class of quantum symmetries  $\Theta$  (see next section for definitions and main properties). Separable states are such that all generalized concurrences are equal to zero and A. Uhlmann proved that the converse is true for the case of density matrices of rank 1 (pure states). He stated that it is ‘unlikely’ that this requirement can be dropped and we will show in this paper that this is indeed the case. Generalized concurrences give however an additional test of entanglement, that is, if we can find a generalized concurrence,  $C_\Theta$ , such that  $C_\Theta(\rho) \neq 0$ , then  $\rho$  is entangled. In this note, we consider generalized concurrences in the simplest case not considered in [15], [16]. That is the case of  $2 \times 4$  systems with density matrices of rank 2. We shall see that, as A. Uhlmann thought, even in this simple situation, the test based on generalized concurrences is not necessary and sufficient and there are entangled states that are undetected.

This paper is organized as follows. In Section 2 we give the main definitions concerning generalized concurrences, describe their role in entanglement detection and recall a connection between certain quantum symmetries and Cartan involutions [7], established in [3], [4], which we shall use in the our derivation. Section 3 presents the main result. We consider bipartite systems where the two subsystems have dimensions 2 and 4 respectively and assume that the density matrix has rank 2. Under these assumptions, the main result,

theorem 3, describes the class of density matrices for which all the concurrences are zero. This set is made up of both separable and entangled states. This shows that there are entangled states that cannot be detected using generalized concurrences. Some conclusions are drawn in section 4. Most of the technical proofs, including the proof of theorem 3, are presented in the Appendixes.

## 2 Quantum symmetries, conjugations, generalized concurrences and Cartan involutions

A *quantum symmetry*  $\Theta$  is a map  $\Theta : \mathcal{H} \rightarrow \mathcal{H}$ , where  $\mathcal{H}$  is Hilbert space, of the form

$$\Theta := e^{i\phi}U, \quad \phi \in \mathbf{R}$$

where  $e^{i\phi}$  is a, physically irrelevant, phase factor and  $U$  is either a unitary or anti-unitary operator. An anti-unitary operator  $U$  is defined by the two properties

$$\begin{aligned} \langle U\beta|U\alpha \rangle &= \langle \beta|\alpha \rangle^*, \\ U(c_1|\alpha \rangle + c_2|\beta \rangle) &= c_1^*U|\alpha \rangle + c_2^*U|\beta \rangle, \end{aligned} \quad (2)$$

for each pair of vectors  $|\alpha \rangle$  and  $|\beta \rangle$  in  $\mathcal{H}$  and pair of complex numbers  $c_1$  and  $c_2$ . Property (2) is referred to as *anti-linearity*. A (*skew*) *conjugation* is an anti-unitary quantum symmetry  $\Theta$  satisfying  $(\Theta^2 = -\mathbf{1})$   $\Theta^2 = \mathbf{1}$ , where  $\mathbf{1}$  is the identity operator.

Given a conjugation  $\Theta$ , A. Uhlmann [15] defined a *generalized concurrence* associated with  $\Theta$  as a function on  $\mathcal{H}$ ,  $C_\Theta(|\psi \rangle)$ , given by

$$C_\Theta(|\psi \rangle) := |\langle \psi|\Theta|\psi \rangle|. \quad (3)$$

This definition is extended to states represented by a general density matrix  $\rho$  using the *convex roof* procedure (cf. [15]). This means that  $C_\Theta(\rho)$  is defined as

$$C_\Theta(\rho) := \min \sum_j \mu_j C_\Theta(|\psi_j \rangle), \quad (4)$$

where the minimum is taken among all the possible decompositions of  $\rho$  as

$$\rho = \sum_j \mu_j |\psi_j \rangle \langle \psi_j|, \quad \mu_j > 0, \quad \sum_j \mu_j = 1. \quad (5)$$

Associated with a quantum symmetry  $\Theta$  is a super-operator  $\theta$ , mapping linear Hermitian operators to linear Hermitian operators, defined as

$$\theta(\rho) := \Theta\rho\Theta^{-1}. \quad (6)$$

It can be easily verified that  $\theta$  is linear, positive and trace preserving map (in fact,  $\theta(\rho)$  has the same eigenvalues as  $\rho$ ). While  $\Theta$  determines  $\theta$  according to formula (6) specification of  $\theta$  on all linear Hermitian operators determines  $\Theta$  up to a phase factor [5]. If  $\Theta$  is a conjugation or a skew-conjugation, then  $\Theta^{-1} = \pm\Theta$  and  $\theta^2$  is equal to the identity operator. A. Uhlmann [15] gave a general method to calculate generalized concurrences, that is to find the minimum in (4), in terms of the superoperator  $\theta$  defined in (6). We collect this result in the following theorem.

**Theorem 1** [15] *Assume  $\Theta$  is a conjugation and consider the matrix*

$$\rho^{\frac{1}{2}}\theta(\rho)\rho^{\frac{1}{2}} := \rho^{\frac{1}{2}}\Theta\rho\Theta\rho^{\frac{1}{2}}, \quad (7)$$

*which is positive semi-definite. If  $\lambda_{max}$  is its largest eigenvalue and  $\lambda_1, \dots, \lambda_{n-1}$  are the remaining (possibly repeated) eigenvalues, then*

$$C_\Theta(\rho) = \max\{0, \sqrt{\lambda_{max}} - \sum_{j=1}^{n-1} \sqrt{\lambda_j}\}. \quad (8)$$

In [3], [4] a connection was recognized between quantum symmetries and *Cartan involutions* used in Cartan classification of symmetric spaces of the Lie group  $SU(n)$ . This connection is useful to study the dynamics of generalized concurrences as well as to parametrize conjugations and skew-conjugations. We explain this next.

Consider a quantum symmetry  $\Theta$  and the corresponding  $\theta$  defined as in (6). Assume  $\theta$  is such that  $\theta^2$  is equal to the identity operator. Then  $\Theta$  is a conjugation, a skew-conjugation or a unitary symmetry whose square is a multiple of the identity. We shall call a quantum symmetry with this property a *Cartan involutory symmetry* for reasons that will be apparent shortly. Consider the space of Hermitian operators on an  $n$ -dimensional Hilbert space. As  $u(n)$  denotes the space of  $n \times n$  skew-Hermitian matrices, we denote by  $iu(n)$  the space of  $n \times n$  Hermitian matrices. This space, when equipped with the anti-commutator operation,  $\{A, B\} := AB + BA$ , is a Jordan algebra and  $\theta$  defined in (6) is a Jordan algebra isomorphism  $iu(n) \rightarrow iu(n)$  satisfying  $\theta^2 = \mathbf{1}$ , with  $\mathbf{1}$  the identity operator. Let  $\mathcal{P}$  and  $\mathcal{K}$  subspaces of  $u(n)$  such that  $i\mathcal{P}$  and  $i\mathcal{K}$  are the  $+1$  and  $-1$ , eigenspaces of  $\theta$ . The map  $\tilde{\theta}$  on  $u(n)$  defined by

$$\tilde{\theta}(A) := i\theta(iA). \quad (9)$$

is a Lie algebra isomorphism of  $u(n)$  and it is such that  $\mathcal{K}$  and  $\mathcal{P}$  are the  $+1$  and  $-1$  eigenspaces of  $\tilde{\theta}$ . Moreover  $\tilde{\theta}^2 = \mathbf{1}$ . A Lie algebra isomorphism with this property is called a *Cartan involution* [7]. Therefore, there exists a one to one correspondence given by formula (9) between Cartan involutions and Cartan involutory symmetries. A Cartan involution  $\tilde{\theta}$  induces a Cartan decomposition of the Lie algebra  $u(n)$  which means that the associated subspaces  $\mathcal{K}$  and  $\mathcal{P}$  satisfy the commutation relations<sup>1</sup>

$$[\mathcal{K}, \mathcal{K}] \subseteq \mathcal{K}, \quad [\mathcal{K}, \mathcal{P}] \subseteq \mathcal{P}, \quad [\mathcal{P}, \mathcal{P}] \subseteq \mathcal{K}. \quad (10)$$

According to Cartan decomposition theorem [7] to the Lie algebra decomposition (10) there corresponds a decomposition of the Lie group  $U(n)$ , in that every element  $X$  of  $U(n)$  can be written as  $X = KP$  where  $K$  is in the connected Lie group,  $e^{\mathcal{K}}$ , associated with  $\mathcal{K}$  (which from the first one of (10) is a Lie subalgebra of  $u(n)$ ) and  $P$  is the exponential of an element in  $\mathcal{P}$ . The quotient space  $U(n)/e^{\mathcal{K}}$  is called a *symmetric space* of  $U(n)$ . Cartan has classified all the symmetric spaces of  $U(n)$  and therefore all the decompositions of the type (10) and all the Cartan involutions [7]. From the correspondence (9), this gives a classification of all the Cartan involutory symmetries. There are three types of Cartan involutions labeled by **AI**, **AII** and **AIII** which correspond, respectively to conjugations, skew-conjugations and unitary symmetries. In particular, it follows, using Cartan parametrization (cf. Tables 5.1 and 5.2 in [3]), that up to a phase factor, every conjugation can be written as  $\Theta_I(|\psi\rangle) = TT^T|\bar{\psi}\rangle$  while every skew-conjugation can be written as

$$\Theta_{II}(|\psi\rangle) = TJT^T|\bar{\psi}\rangle. \quad (11)$$

In these formulas and in the following  $|\bar{\psi}\rangle$  ( $\bar{\rho}$ ) denotes the complex conjugate of the vector  $|\psi\rangle$  (of a matrix  $\rho$ ),  $J$  is the matrix  $J := \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$ , where  $\mathbf{1}$  is the  $\frac{n}{2} \times \frac{n}{2}$  (assuming  $n$  even) identity and  $T$  is an arbitrary (parameter) special unitary matrix, i.e. a matrix in  $SU(n)$ . We now turn to the application of this theory to entanglement.

When dealing with bipartite systems, whose Hilbert space  $\mathcal{H}$  is the tensor product of two Hilbert spaces  $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$ , it is natural to construct quantum symmetries on  $\mathcal{H}$  as tensor products of symmetries on  $\mathcal{H}_A$  and  $\mathcal{H}_B$ .<sup>2</sup> Consider two spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$  both with even dimension. A. Uhlmann [15] considers conjugations  $\Theta$  constructed as tensor products of skew-conjugations,  $\Theta_A$  and  $\Theta_B$ . Using the characterization of skew-conjugations (11) it is straightforward to prove the following.

**Theorem 2** *If  $\rho$  is a separable state, then  $C_\Theta(\rho) = 0$  for every conjugation  $\Theta = \Theta_A \otimes \Theta_B$ , with  $\Theta_{A,B}$  skew-conjugations on  $\mathcal{H}_{A,B}$*

<sup>1</sup>The viceversa is also true. That is, if one has a Cartan decomposition of the form (10) then one can define a Cartan involution as a Lie algebra isomorphism having  $\mathcal{K}$  and  $\mathcal{P}$  as the  $+1$  and  $-1$  eigenspaces, respectively.

<sup>2</sup>The tensor product of two anti-linear (linear) operators  $\Theta_A \otimes \Theta_B$  is defined on product vectors  $|\psi_A\rangle \otimes |\psi_B\rangle$  as  $\Theta_A \otimes \Theta_B(|\psi_A\rangle \otimes |\psi_B\rangle) := \Theta_A(|\psi_A\rangle) \otimes \Theta_B(|\psi_B\rangle)$  and then extended by anti-linearity (linearity) for linear combinations of product vectors.

**Proof** We use the general form of a skew-conjugation (11) and define  $\Theta_{A,B}(|\psi_{A,B}\rangle) := T_{A,B} J T_{A,B}^T \overline{|\psi_{A,B}\rangle}$ , with general matrices  $T_A \in SU(n_A)$  and  $T_B \in SU(n_B)$ . Using the decomposition (1) for  $\rho$  with the definition (4) for the generalized concurrence  $C_\Theta$ , we have

$$0 \leq C_\Theta(\rho) \leq \sum_j \mu_j C_\Theta(|\psi_{Aj}\rangle \otimes |\psi_{Bj}\rangle) = 0.$$

The last equality is due to the fact that, with  $\Theta := \Theta_A \otimes \Theta_B$ , for each  $j$ , we have from (3),

$$\begin{aligned} C_\Theta(|\psi_{Aj}\rangle \otimes |\psi_{Bj}\rangle) &= |\langle \psi_{Aj} | \otimes \langle \psi_{Bj} | (\Theta_A \otimes \Theta_B) | \psi_{Aj}\rangle \otimes |\psi_{Bj}\rangle| = \\ &= |\langle \psi_{Aj} | \otimes \langle \psi_{Bj} | T_A J T_A^T \otimes T_B J T_B^T \overline{|\psi_{Aj}\rangle} \otimes \overline{|\psi_{Bj}\rangle}| = \left| \langle T_A^\dagger \psi_{Aj} | J | T_A^\dagger \psi_{Aj} \rangle \right| \times \left| \langle T_B^\dagger \psi_{Bj} | J | T_B^\dagger \psi_{Bj} \rangle \right| = 0, \end{aligned}$$

as both factors in the last expression are zero.  $\square$

This theorem gives a method to detect entanglement. If there exists a  $\Theta$  such that  $C_\Theta(\rho) > 0$ , then  $\rho$  is entangled. To calculate  $C_\Theta$  one can use formula (8). If  $C_\Theta(\rho) = 0$  for every  $\Theta$ , then it may be separable or entangled (as we shall see below) and, in the latter case, entanglement is not detected by this method.

In the special case of two qubits, i.e.,  $n_A = n_B = 2$ ,  $T_A$  and  $T_B$  are general matrices in  $SU(2)$  and, for every matrix  $T$  in  $SU(2)$ ,

$$T J T^T = J. \quad (12)$$

As a consequence of formula (11) there is only one (generalized) concurrence corresponding to the conjugation  $\Theta(|\psi\rangle) = J \otimes J \overline{|\psi\rangle}$ . This is the concurrence originally considered by Hill and Wootters in [8], [16]. In this case the converse of theorem 2 holds, that is, if the (only) concurrence is zero the state is separable. It was proven in [15] that the converse also holds when  $\rho$  is a pure state. In the next section, we settle in the negative the question of whether the converse holds in general.

### 3 Main Result

From now on, we shall consider only, the case of a state  $\rho$  on a Hilbert space  $\mathcal{H}_A \otimes \mathcal{H}_B$  with  $\dim \mathcal{H}_A := n_A = 2$  and  $\dim \mathcal{H}_B := n_B = 4$  and  $\rho$  of rank 2, although some of things we shall say can be extended without difficulties to the general case. In the following we shall also denote by  $J_{2m}$  the  $2m \times 2m$  matrix  $J_{2m} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$  where  $\mathbf{1}$  is the  $m \times m$  identity. If  $\Theta$  is the tensor product of two skew-conjugations, then, using (11),  $\Theta$  has the form  $\Theta|\psi\rangle = T_A J_2 T_A^T \otimes T_B J_4 T_B^T \overline{|\psi\rangle}$  with  $T_A \in SU(2)$  and  $T_B \in SU(4)$ . From the above recalled property (12) of  $T_A \in SU(2)$  we have that defining

$$M := J_2 \otimes T J_4 T^T, \quad (13)$$

with  $T \in SU(4)$ , every conjugation  $\Theta$  which is tensor product of two skew-conjugations can be written as

$$\Theta(|\psi\rangle) = M \overline{|\psi\rangle} \quad (14)$$

and by varying  $T \in SU(4)$  we obtain all of such products. A state  $\rho$  which has zero concurrence  $C_\Theta$  for all such  $\Theta$ 's will be called a *Zero Concurrence (ZC)-state*. Therefore from theorem 2 it follows that separable states are ZC-states. In theorem 3, we shall see that the set of ZC-states is made up of two nonempty subsets containing respectively only separable and only entangled states.

All the properties of  $\rho$  which are of interest to us (separability, PPT and ZC) are invariant under local transformations, i.e., under transformations of the form  $\rho \rightarrow (X_1 \otimes X_2) \rho (X_1^\dagger \otimes X_2^\dagger)$  with  $X_1 \in SU(2)$  and  $X_2 \in SU(4)$ . Namely, we have the following.

**Proposition 1** *For every  $X_1 \in SU(2)$  and  $X_2 \in SU(4)$*

1.  $\rho$  is separable if and only if  $(X_1 \otimes X_2) \rho (X_1^\dagger \otimes X_2^\dagger)$  is separable.
2.  $\rho$  is PPT if and only if  $(X_1 \otimes X_2) \rho (X_1^\dagger \otimes X_2^\dagger)$  is PPT.

3.  $\rho$  is  $ZC$  if and only if  $(X_1 \otimes X_2)\rho(X_1^\dagger \otimes X_2^\dagger)$  is  $ZC$ .

**Proof.** The first two properties are obvious. Now, assume that  $\rho$  is a  $ZC$ -state and let  $\Theta$  a general conjugation (14) corresponding to a matrix  $M$  as in (13). Let  $\tilde{\Theta}$  a conjugation corresponding to matrix  $\tilde{M} := J_2 \otimes X_4^\dagger T J_4 T^T \overline{X_4}$ . Since  $\rho$  is  $ZC$ ,  $C_{\tilde{\Theta}}(\rho) = 0$ . In particular, there exists a decomposition of  $\rho$  as in (5) such that, for every  $j$ ,

$$0 = C_{\tilde{\Theta}}(|\psi_j\rangle) = \left| \langle \psi_j | J_2 \otimes X_4^\dagger T J_4 T^T \overline{X_4} | \overline{\psi_j} \rangle \right| = \left| \langle \psi_j | X_2^\dagger \otimes X_4^\dagger (J_2 \otimes T J_4 T^T) \overline{X_2} \otimes \overline{X_4} | \overline{\psi_j} \rangle \right|. \quad (15)$$

However the last term of (15) is  $C_{\Theta}(X_2 \otimes X_4 |\psi_j\rangle)$  and  $\{\mu_j, X_2 \otimes X_4 |\psi_j\rangle\}$  give a decomposition of  $(X_2 \otimes X_4)\rho(X_2^\dagger \otimes X_4^\dagger)$ . Since  $\Theta$  is arbitrary  $(X_2 \otimes X_4)\rho(X_2^\dagger \otimes X_4^\dagger)$  is  $ZC$  as well. The converse follows immediately from the fact that  $X_2$  and  $X_4$  are arbitrary.  $\square$

The previous property suggests to place  $\rho$  in a *canonical form* using only local transformations, without loss of generality. We shall describe this canonical form next. Since  $\rho$  has rank 2, we write it in terms of its eigenvectors corresponding to nonzero eigenvalues as

$$\rho = \lambda |\psi_1\rangle\langle\psi_1| + (1 - \lambda) |\psi_2\rangle\langle\psi_2|, \quad (16)$$

with  $0 < \lambda < 1$ . We assume that at least one between  $|\psi_1\rangle$  and  $|\psi_2\rangle$  is an entangled pure state.<sup>3</sup> If both are  $|\psi_1\rangle$  and  $|\psi_2\rangle$  are separable, then  $\rho$  is also separable and therefore it is  $ZC$  and  $PPT$ . Excluding this case, we assume  $|\psi_1\rangle$  entangled. Using Schmidt decomposition theorem (cf. Theorem 2.7 in [13]) we choose orthonormal bases  $\{|a_{1,2}\rangle\}$  of  $\mathcal{H}_A$  and  $\{|b_{1,2,3,4}\rangle\}$  of  $\mathcal{H}_B$  such that  $|\psi_1\rangle = q_1 |a_1, b_1\rangle + q_6 |a_2, b_2\rangle$ , with both  $q_1$  and  $q_6$  real and nonnegative. Moreover since  $|\psi_1\rangle$  is entangled, both  $q_1$  and  $q_6$  are strictly positive. Using these bases, we write  $|\psi_2\rangle := \sum_{j=1,2, k=1,\dots,4} r_{jk} |a_j, b_k\rangle$ . We use a local transformation of the form  $\mathbf{1} \otimes X$  where  $X \in SU(4)$  acts as the identity on the subspace spanned by  $|b_1\rangle$  and  $|b_2\rangle$ , to set the coefficient  $r_{14}$  to zero without changing  $|\psi_1\rangle$ . Finally, since  $|\psi_2\rangle$  (and  $|\psi_1\rangle$ ) is defined up to an overall phase factor we assume  $r_{11}$  real and nonnegative. Since  $\langle\psi_1|\psi_2\rangle = 0 = q_1 r_{11} + q_6 r_{22}$  which forces  $r_{22}$  to be real and non-positive. In conclusion,  $\rho$  in (16) is such that either both  $|\psi_1\rangle$  and  $|\psi_2\rangle$  are separable (and it is therefore separable) or it can be transformed with local transformations into a canonical form where the  $|\psi_1\rangle$  and  $|\psi_2\rangle$  coordinates are, respectively,  $\psi_1 := (q_1, 0, 0, 0, 0, q_6, 0, 0)^T$ ,  $\psi_2 := (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8)^T$  with  $q_1 > 0$  and  $q_6 > 0$ ,  $p_1 \geq 0$ ,  $p_6 \leq 0$ ,  $p_4 = 0$  and  $p_1 q_1 + p_6 q_6 = 0$ . This is the canonical form we shall refer to in the sequel.

We now state a fact which is a special case of a general result proven in [11], [12].

**Proposition 2** Assume  $\rho$  is a  $2 \times 4$  state with rank 2. Then  $\rho$  is separable if and only if it is  $PPT$ .

The above proposition says that the  $PPT$  test characterizes completely separable and entangled states in the  $2 \times 4$ , rank 2, case. Some more information we shall use is given in the following Lemma.

**Lemma 1** Let  $\rho$  be a state in canonical form. Then  $\rho$  is  $PPT$  and therefore separable if and only if it has the form

$$\rho = \begin{pmatrix} \rho_{11} & 0 & \rho_{12} & 0 \\ 0 & 0 & 0 & 0 \\ \rho_{12}^\dagger & 0 & \rho_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (17)$$

where the  $4 \times 4$  matrix

$$\tilde{\rho} := \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^\dagger & \rho_{22} \end{pmatrix} \quad (18)$$

is separable as a two qubit state.

We give the proof of this Lemma in Appendix C. This also gives an alternative proof of Proposition 2.

We are now ready to state our main result which describes completely the set of,  $2 \times 4$ ,  $ZC$ -states of rank two. The proof is given in Appendix B while auxiliary results are presented in Appendix A.

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<sup>3</sup>There are several general methods to check that a pure bipartite state is entangled. An example is given by the entropy cf., e.g., [13].

**Theorem 3** A  $2 \times 4$ , rank 2, state  $\rho$  is a ZC-state if and only if it is in one of the following two disjoint classes.

- The class ZCS ( $S$  stands for separable) which is defined as containing states of the form (16) with  $|\psi_1\rangle$  and  $|\psi_2\rangle$  separable along with PPT-states which can be written in canonical form as in (17).
- States of the form (16) which can be written in canonical form with

$$\lambda = \frac{1}{2}, \quad |\psi_1\rangle = q_1|a_1, b_1\rangle + q_6|a_2, b_2\rangle, \quad |\psi_2\rangle = q_1|a_1, b_3\rangle + q_6e^{i\phi}|a_2, b_4\rangle \quad \phi \in \mathbb{R}. \quad (19)$$

These states will be called ZCE-states ( $E$  stands for entangled).

## 4 Conclusions

Several extensions of the concurrence originally defined by Hill and Wootters [8], [16] for the  $2 \times 2$  case have been proposed in the literature (see e.g., [10]). In few cases a direct physical in terms of probability for the measurements of appropriate observables has been indicated [2]. However in most cases, a direct physical interpretation is missing. The generalized concurrences considered here are the ones studied in [15]. They are functions constructed through anti-linear operators (symmetries). As observed in [15] these operators are intrinsically non-local as there is no way to tensor them with the identity, that is, to apply them to a part of the system by leaving the other unchanged. Using these operators, a family of functions can be constructed which are all zero if the state is separable. The question then arises on whether these functions provide a complete test to detect entanglement. In this note, we have given a negative answer to this question. The PPT test is necessary and sufficient for entanglement of  $2 \times 4$  states of rank 2 [11], [12]. Generalized concurrences can be used to detect entanglement, but in this case they do not detect entanglement for a class of states (ZCE states) we have described in theorem 3.

In spite of this negative result, we believe generalized concurrences are still worth further investigation. In particular, it is an open question whether for higher dimensional problems, and-or higher rank, generalized concurrences may detect entanglement of PPT states.

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## References

- [1] R.T. Browne, *Introduction to the Theory of Determinants and Matrices*, University of North Carolina Press, Chapel Hill, NC, 1958.
- [2] M. A. Cirone, Entanglement correlations, Bell inequalities and the concurrence, *Physics Letters A* 339 (2005) 269-274.
- [3] D. D'Alessandro, *Introduction to Quantum Control and Dynamics*, CRC Press, Boca Raton, FL, 2007.
- [4] D. D'Alessandro and F. Albertini, Quantum symmetries and Cartan decompositions in arbitrary dimensions, *J. Phys. A* **40**, 2439-2453 (2007).
- [5] A. Galindo and P. Pascual, *Quantum Mechanics 1*, Texts and Monographs in Physics, Springer-Verlag, Heidelberg, 1990.
- [6] F.R. Gantmacher, *Matrix Theory*, vol. 1, Chelsea, New York, 1959, p. 307.
- [7] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, 1978.
- [8] S. Hill and W. Wootters, Entanglement of a pair of quantum bits, *Phys. Rev. Lett.* **78**, 5022-5025 (1997).

- [9] M. Horodecki, P. Horodecki, and R. Horodecki, Separability of mixed states: necessary and sufficient conditions, *Phys. Lett. A* **223**, 1-8 (1996).
- [10] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, Quantum entanglement, arXiv:quant-ph/0702225.
- [11] P. Horodecki, M. Lewenstein, G. Vidal, and I. Cirac, Operational criterion and constructive checks for the separability of low rank density matrices, *Phys. Rev. A* **62**, 032310 (2000).
- [12] B. Kraus, J. I. Cirac, S. Karnas, and M. Lewenstein, Separability in  $2 \times N$  composite quantum systems, *Phys. Rev. A* **61**, 062302 (2000).
- [13] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2000.
- [14] A. Peres, Separability criterion for density matrices, *Phys. Rev. Lett.* **77**, 1413-1415 (1996).
- [15] A. Uhlmann, Fidelity and concurrence of conjugated states, *Phys. Rev. A* **62**, 032307 (2000).
- [16] W. K. Wootters, Entanglement of formation of an arbitrary state of two qubits, *Phys. Rev. Lett.* **80**, 2245-2248 (1998).

## Appendix A: Two Auxiliary Lemmas

The matrix  $M$  in (13) determines the particular generalized concurrence considered.  $ZC$ -states, by definition, have all the concurrences equal to zero. In principle  $M$  depends on 15 parameters since it depends on the matrix  $T$ , which is a general matrix in  $SU(4)$  whose dimension is 15. However, the form of  $M$  can be greatly simplified. Using the Cartan decomposition of type **AII** [7], every  $T \in SU(4)$  can be written as  $T = PK$ , where  $K$  is symplectic and  $P = e^G$  with  $G \in \mathfrak{sp}(2)^{\perp}$ . Matrices in  $\mathfrak{sp}(2)^{\perp}$  have the form

$$G = \begin{pmatrix} A & bJ_2 \\ \bar{b}J_2 & A^T \end{pmatrix}, \quad (20)$$

with  $A$   $2 \times 2$  skew-Hermitian and  $b$  a complex scalar. Since every symplectic matrix  $K$  is by definition such that  $KJ_4K^T = J_4$ , we can rewrite every  $M$  in (13) in the form

$$M = J_2 \otimes e^{Gt} J_4 e^{G^T t}, \quad t \in \mathbf{R}. \quad (21)$$

Defining  $H := 2GJ_4$  and  $\eta := \frac{1}{2}\sqrt{\text{Tr}(HH^\dagger)}$ , the following relations are easily verified:

$$GJ_4 = J_4G^T := \frac{1}{2}H, \quad GH + HG^T = -\eta^2 J_4. \quad (22)$$

The first lemma of this appendix gives a simplified expression for  $M$ .

**Lemma 2** *For any  $M$  in (13),  $M \neq J_2 \otimes J_4$ , there exists a  $G$  (20) and  $H = 2GJ_4$  and  $\eta = \frac{1}{2}\sqrt{\text{Tr}(HH^\dagger)} \neq 0$  such that*

$$M = J_2 \otimes \left( \cos(\eta t) J_4 + \frac{\sin(\eta t)}{\eta} H \right). \quad (23)$$

**Proof.** If  $M \neq J_2 \otimes J_4$  then  $G \neq 0$  and therefore  $H \neq 0$  and  $\eta \neq 0$ . From (21), it is sufficient to prove that

$$F_1(t) := e^{Gt} J_4 e^{G^T t} = \cos(\eta t) J_4 + \frac{\sin(\eta t)}{\eta} H =: F_2(t).$$

The matrix functions  $F_1$  and  $F_2$  are such that  $\dot{F}_1 = GF_1 + F_1G^T$  and  $\dot{F}_2 = GF_2 + F_2G^T$ . The first equation is straightforward, while the second one follows from the relations in (22). Since  $F_1$  and  $F_2$  satisfy the same

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<sup>4</sup>The Lie algebra  $\mathfrak{sp}(m)$  is defined as the one of skew-Hermitian  $2m \times 2m$  matrices  $A$ , satisfying  $AJ_{2m} + J_{2m}A^T = 0$ . The orthogonal complement  $\mathfrak{sp}(m)^{\perp}$  in  $u(2m)$  is taken with respect to the inner product  $(A, B) := \text{Trace}(AB^\dagger)$ .

differential equations and are equal at  $t = 0$  they are the same for every  $t$ .  $\square$

**Remark.** For  $\eta = 0$ ,  $H$  and  $G$  are equal to zero and  $M$  becomes

$$M = J_2 \otimes J_4. \quad (24)$$

This expression can be obtained as the limit of (23) when  $\eta \rightarrow 0$ .  $\square$

In the next result, we consider a general symmetric  $2 \times 2$  complex matrix  $C = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$  and a diagonal matrix  $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1 - \lambda \end{pmatrix}$ , with  $0 < \lambda < 1$ . We are interested in the eigenvalues of the positive semidefinite matrix  $B := \sqrt{\Lambda} C \Lambda C^\dagger \sqrt{\Lambda}$ ,  $\lambda_{max}$  and  $\lambda_{min}$ , and, in particular, in whether or not they are equal. The following lemma gives necessary and sufficient conditions for this to happen.

**Lemma 3** *The two eigenvalues of  $B$  defined above,  $\lambda_{max}$  and  $\lambda_{min}$ , are equal if and only if the following two conditions are verified.*

$$(i) \quad \lambda|\alpha| = (1 - \lambda)|\gamma|;$$

$$(ii) \quad \alpha\gamma\bar{\beta}^2 \leq 0.$$

**Proof.** The eigenvalues  $\lambda_{max}$  and  $\lambda_{min}$  are equal if and only if  $(\lambda_{max} - \lambda_{min})^2 = (\text{Tr}(B))^2 - 4 \det B = 0$ . Using the explicit expression of  $B$ ,

$$B = \begin{pmatrix} \frac{\lambda^2|\alpha|^2 + \lambda(1 - \lambda)|\beta|^2}{\sqrt{\lambda(1 - \lambda)}(\lambda\bar{\alpha}\beta + (1 - \lambda)\bar{\beta}\gamma)} & \frac{\sqrt{\lambda(1 - \lambda)}(\lambda\alpha\bar{\beta} + (1 - \lambda)\beta\bar{\gamma})}{\lambda(1 - \lambda)|\beta|^2 + (1 - \lambda)^2|\gamma|^2} \end{pmatrix},$$

we calculate

$$\begin{aligned} (\text{Tr}(B))^2 - 4 \det B &= (\lambda^2|\alpha|^2 + 2\lambda(1 - \lambda)|\beta|^2 + (1 - \lambda)^2|\gamma|^2 + 2\lambda(1 - \lambda)|\alpha\gamma - \beta^2|) \\ &\quad \cdot (\lambda^2|\alpha|^2 + 2\lambda(1 - \lambda)|\beta|^2 + (1 - \lambda)^2|\gamma|^2 - 2\lambda(1 - \lambda)|\alpha\gamma - \beta^2|). \end{aligned}$$

The first factor in this expression is zero only if  $\alpha = \beta = \gamma = 0$ . If this is not the case, we must have

$$\lambda^2|\alpha|^2 + 2\lambda(1 - \lambda)|\beta|^2 + (1 - \lambda)^2|\gamma|^2 = 2\lambda(1 - \lambda)|\alpha\gamma - \beta^2|. \quad (25)$$

Since this equation is trivially verified also in the special case  $\alpha = \beta = \gamma = 0$ , it is necessary and sufficient to have  $\lambda_{max} = \lambda_{min}$ . Equation (25) can be written in the simpler form (i) and (ii) proceeding as follows.

By the triangular inequality, we have that

$$2\lambda(1 - \lambda)|\alpha\gamma - \beta^2| \leq 2\lambda(1 - \lambda)(|\alpha\gamma| + |\beta|^2)$$

and thus

$$\lambda^2|\alpha|^2 + (1 - \lambda)^2|\gamma|^2 - 2\lambda(1 - \lambda)|\alpha\gamma| \leq 0.$$

But the l.h.s. of the last inequality is equal to  $(\lambda|\alpha| - (1 - \lambda)|\gamma|)^2$  and thus it is positive. Hence,  $\lambda|\alpha| - (1 - \lambda)|\gamma| = 0$ , i.e., (i) is satisfied. If we insert this condition in (25), we get

$$\lambda^2|\alpha|^2 + \lambda(1 - \lambda)|\beta|^2 = \lambda(1 - \lambda)|\alpha\gamma - \beta^2|,$$

where  $\lambda^2|\alpha|^2$  can be rewritten as  $\lambda(1 - \lambda)|\alpha\gamma|$ , because of (i). We then divide both sides of the equation by  $\lambda(1 - \lambda)$  (since  $0 < \lambda < 1$ , we have  $\lambda(1 - \lambda) \neq 0$ ). We obtain  $|\alpha\gamma| + |\beta|^2 = |\alpha\gamma - \beta^2|$ , which is equivalent to condition (ii).

Conversely, if conditions (i) and (ii) are satisfied, then

$$\lambda^2|\alpha|^2 + 2\lambda(1 - \lambda)|\beta|^2 + (1 - \lambda)^2|\gamma|^2 - 2\lambda(1 - \lambda)|\alpha\gamma - \beta^2| = 2\lambda(1 - \lambda)(|\alpha\gamma| + |\beta|^2 - |\alpha\gamma - \beta^2|) = 0,$$

i.e., equation (25).  $\square$



## Appendix B: Proof of theorem 3

The fact that states of the form (16) with  $|\psi_1\rangle$  and  $|\psi_2\rangle$  both separable are  $ZC$  and separable follows from theorem 2. Let us therefore consider a state in canonical form without loss of generality (cf. Proposition 1). A state is a  $ZC$ -state if and only if the matrix  $\rho^{\frac{1}{2}}\theta(\rho)\rho^{\frac{1}{2}}$  in (7) of theorem 1 has two coinciding eigenvalues, for every  $\Theta$  in (14) with  $M$  in (13), (23). By writing  $\rho$  as  $U\tilde{\Lambda}U^\dagger$ , with  $U$  unitary and  $\tilde{\Lambda}$  equal to zero except for the first two entries on the diagonal which are equal to  $\lambda$  and  $1-\lambda$ , it can be seen that the eigenvalues of  $\rho^{\frac{1}{2}}\theta(\rho)\rho^{\frac{1}{2}}$  are the same as the eigenvalues of a  $2 \times 2$  matrix of the form  $B$  considered in Lemma 3.<sup>5</sup> In this case  $\lambda$  and  $1-\lambda$  are the eigenvalues of  $\rho$  as in (16) and  $\alpha = \langle\psi_1|M|\overline{\psi_1}\rangle$ ,  $\beta = \langle\psi_1|M|\overline{\psi_2}\rangle$ ,  $\gamma = \langle\psi_2|M|\overline{\psi_2}\rangle$ , with  $|\psi_1\rangle$  and  $|\psi_2\rangle$  also as in (16) and for every  $M$  in (13). In the following discussion we shall always assume, without loss of generality, that the state  $\rho$  is in canonical form.

If we calculate the explicit form for  $\alpha$  and  $\gamma$ , using the expression for  $M$  in (23), (24), (20), (22), we obtain

$$\alpha = -4b \frac{\sin \eta t}{\eta} q_1 q_6, \quad (26)$$

$$\gamma = 4 \frac{\sin(\eta t)}{\eta} (\bar{w}_2^T J_2 \bar{w}_4 - b \bar{w}_1^T J_2 \bar{w}_3) + 2 \operatorname{Tr} \left[ \left( \cos(\eta t) \mathbf{1} + 2 \frac{\sin(\eta t)}{\eta} A \right) (\bar{w}_4 \bar{w}_1^T - \bar{w}_2 \bar{w}_3^T) \right], \quad (27)$$

where we have partitioned  $|\psi_2\rangle$  as  $|\psi_2\rangle := (w_1^T, w_2^T, w_3^T, w_4^T)^T$  for 2-dimensional vectors  $w_j$ ,  $j = 1, \dots, 4$ . If  $\rho$  is a  $ZC$ -state, equation (i) of Lemma 3 has to hold with  $\alpha$  and  $\gamma$  for every skew-Hermitian zero trace matrix  $A$ , every real  $t$ , and every complex number  $b$ . In particular, by setting  $b = 0$  and varying  $t$  and  $A$ , we obtain that it must be

$$w_4 w_1^T = w_2 w_3^T, \quad (28)$$

and the second term in the r.h.s. of (27) is zero. Inserting this constraint in (i) of Lemma 3, we have that for every complex number  $b$

$$\frac{\lambda}{1-\lambda} |b| q_1 q_6 = |b w_2^T J_2 w_4 - \bar{b} w_1^T J_2 w_3|$$

must hold. For this to be verified one and only one between  $w_2^T J_2 w_4$  and  $w_1^T J_2 w_3$  must be different from zero and equal to  $\frac{\lambda}{1-\lambda} q_1 q_6$  in absolute value. Let us indicate by  $ZCS$ ,  $ZC$ -states such that  $w_1^T J_2 w_3 \neq 0$  and by  $ZCE$  its complement in the set of  $ZC$ -states. If a state is  $ZCS$ , multiplying (28) on the right by  $J_2 w_1$  and using the fact that  $w_3^T J_2 w_1 \neq 0$  but  $w_1^T J_2 w_1 = 0$ , we obtain  $w_2 = 0$ . Analogously, multiplying by  $J_2 w_3$  we obtain  $w_4 = 0$ . In a similar fashion for  $ZCE$  states, we obtain  $w_1 = 0$  and  $w_3 = 0$ . Summarizing, if a state is  $ZC$ , it has to be of the form  $ZCS$  with

$$w_2 = w_4 = 0, \quad |w_1^T J_2 w_3| = \frac{\lambda}{1-\lambda} q_1 q_6, \quad (29)$$

or of the form  $ZCE$  with

$$w_1 = w_3 = 0, \quad |w_2^T J_2 w_4| = \frac{\lambda}{1-\lambda} q_1 q_6, \quad (30)$$

In order to analyze the implications of the condition (ii) of Lemma 3, we write  $\beta$  in the two cases,  $ZCS$  and  $ZCE$ , and denote it by  $\beta_S$  and  $\beta_E$ , respectively. With  $v_1 = (q_1, 0)^T$  and  $v_2 = (0, q_6)^T$ , we obtain

$$\beta_S = 2b \frac{\sin(\eta t)}{\eta} (-v_1^T J_2 \bar{w}_3 + v_2^T J_2 \bar{w}_1), \quad (31)$$

$$\beta_E = v_1^T \left( \cos(\eta t) \mathbf{1} + 2 \frac{\sin(\eta t)}{\eta} A \right) \bar{w}_4 - v_2^T \left( \cos(\eta t) \mathbf{1} + 2 \frac{\sin(\eta t)}{\eta} A \right) \bar{w}_2. \quad (32)$$

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<sup>5</sup>We have (cf. (14), (6))  $\rho^{\frac{1}{2}}\theta(\rho)\rho^{\frac{1}{2}} = \rho^{\frac{1}{2}}M\bar{\rho}M^\dagger\rho^{\frac{1}{2}}$ , and, therefore,  $\rho^{\frac{1}{2}}\theta(\rho)\rho^{\frac{1}{2}} = U\tilde{\Lambda}^{\frac{1}{2}}U^\dagger M\bar{U}\tilde{\Lambda}U^T M^\dagger U\tilde{\Lambda}^{\frac{1}{2}}U^\dagger$ . If we denote by  $\tilde{C}$  the symmetric matrix  $U^\dagger M\bar{U}$ , the eigenvalues of  $\rho^{\frac{1}{2}}\theta(\rho)\rho^{\frac{1}{2}}$  are the same as the eigenvalues of  $\tilde{\Lambda}^{\frac{1}{2}}\tilde{C}\tilde{\Lambda}^{\frac{1}{2}}$ . Calling  $\Lambda$  ( $C$ ) the upper  $2 \times 2$  block of the matrix  $\tilde{\Lambda}$  ( $\tilde{C}$ ), since all the other entries of  $\tilde{\Lambda}$  are zeros, it follows that the nonzero eigenvalues are the same as the ones of a matrix of the form  $B$  considered in Lemma 3.

Let us consider the case of *ZCE*-states first. Inserting (30) and (28) in (27) and using  $\beta_E$  in (32) for  $\beta$ , we obtain from condition (ii)

$$-16q_1q_6|b|^2\frac{\sin^2(\eta t)}{\eta^2}\bar{w}_2^T J_2 \bar{w}_4 \left[ v_1^T \left( \cos(\eta t)\mathbf{1} + 2\frac{\sin(\eta t)\bar{A}}{\eta} \right) w_4 - v_2^T \left( \cos(\eta t)\mathbf{1} + 2\frac{\sin(\eta t)\bar{A}}{\eta} \right) w_2 \right]^2 \leq 0. \quad (33)$$

This expression has to hold for every skew-Hermitian matrix  $A$ , every  $t$ , and every  $\eta \neq 0$ . Setting  $A = 0$  and recalling the definition of the  $v_{1,2}$  and  $w_{2,4}$  vectors, and the fact that  $p_4 = 0$ , we obtain  $\bar{p}_3\bar{p}_8p_7^2 \geq 0$ . Setting  $A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$  and  $\cos(\eta t) = 0$ , we obtain  $-\bar{p}_3\bar{p}_8p_7^2 \geq 0$ , which shows that  $p_7 = 0$ , since  $p_3p_8 = w_2^T J_2 w_4 \neq 0$ . Using this to simplify (33), we find that  $p_3$  and  $p_8$  must be such that, for every complex number  $c$

$$\bar{p}_3\bar{p}_8(cq_1p_8 + \bar{c}q_6p_3)^2 \geq 0.$$

It is easily seen that this is the case if and only if  $q_1|p_8| = q_6|p_3|$ . Combining this with (30) and the fact that  $\|\psi_2\| = 1$ , we find that we must have  $\lambda = \frac{1}{2}$  and  $|p_8| = q_6$ ,  $|p_3| = q_1$ . Hence, states of the type *ZCE* must be of the form (19).<sup>6</sup> Consider *ZCS*-states next. In this case, using (29), (26), (27) and (31) with  $b \neq 0$ , we obtain that condition (ii) of Lemma 3 gives  $\bar{w}_1^T J_2 \bar{w}_3 (-v_1^T J_2 w_3 + v_2^T J_2 w_1)^2 \leq 0$ . Writing this in terms of the vectors  $|\psi_1\rangle$  and  $|\psi_2\rangle$ , we obtain the condition

$$(-\bar{p}_2\bar{p}_5 + \bar{p}_1\bar{p}_6)(-q_1p_6 - q_6p_1)^2 \leq 0, \quad (34)$$

which supplements (29) in describing these states. To show that these states correspond to the ones in (17), we consider the two qubit state

$$\tilde{\rho} = \lambda|\tilde{\psi}_1\rangle\langle\tilde{\psi}_1| + (1-\lambda)|\tilde{\psi}_2\rangle\langle\tilde{\psi}_2|$$

with

$$|\tilde{\psi}_1\rangle = (q_1, 0, 0, q_6)^T, \quad |\tilde{\psi}_2\rangle = (p_1, p_2, p_5, p_6)^T,$$

corresponding to (18). We have to show that that  $\tilde{\rho}$  is separable. For this we use the *two qubit* concurrence [16] which gives a necessary and sufficient condition of separability. There is only one concurrence in the two qubit case, which can be defined as in (8), where  $\lambda_{max}$ ,  $\lambda_{1,2,3}$  are the eigenvalues of the matrix

$$\tilde{\rho}^{\frac{1}{2}} J_2 \otimes J_2 \tilde{\rho} J_2 \otimes J_2 \tilde{\rho}^{\frac{1}{2}}.$$

A two qubit state  $\tilde{\rho}$  is separable if and only if the concurrence is zero. Using the fact that the state has rank two and proceeding as for the  $2 \times 4$  case, now with  $M = J_2 \otimes J_2$ , we have that this is verified if and only if both conditions of Lemma 3 are verified, with  $\alpha$ ,  $\beta$ , and  $\gamma$  given now by

$$\alpha = 2q_1q_6, \quad \beta = q_1\bar{p}_6 + q_6\bar{p}_1, \quad \gamma = 2(\bar{p}_1\bar{p}_6 - \bar{p}_2\bar{p}_5).$$

Formula (i) gives the second one of (29) and formula (ii) gives (34).

Summarizing, *ZC*-states must be in one of the classes *ZCS* and *ZCE* of the statement of the theorem. Viceversa, if a state is *ZCS*, it is a separable state and therefore it is a *ZC*-state. If a state is *ZCE*, it is straightforward to verify by plugging (19) in the expressions (26), (27) and (32) that conditions (i) and (ii) of Lemma 3 are verified for every concurrence. This concludes the proof of the theorem.  $\square$

## Appendix C: Proof of Lemma 1

To simplify notations, it is convenient to use  $\alpha_{jk} := (1-\lambda)p_j\bar{p}_k$  with  $j \leq k$  and  $\beta_{jk} := \lambda q_j q_k$ . This way,  $\rho^{TA}$  writes as

$$\rho^{TA} = \begin{pmatrix} \beta_{11} + \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & \bar{\alpha}_{15} & \bar{\alpha}_{25} & \bar{\alpha}_{35} & 0 \\ \bar{\alpha}_{12} & \alpha_{22} & \alpha_{23} & 0 & \beta_{16} + \bar{\alpha}_{16} & \bar{\alpha}_{26} & \bar{\alpha}_{36} & 0 \\ \bar{\alpha}_{13} & \bar{\alpha}_{23} & \alpha_{33} & 0 & \bar{\alpha}_{17} & \bar{\alpha}_{27} & \bar{\alpha}_{37} & 0 \\ 0 & 0 & 0 & 0 & \bar{\alpha}_{18} & \bar{\alpha}_{28} & \bar{\alpha}_{38} & 0 \\ \alpha_{15} & \beta_{16} + \alpha_{16} & \alpha_{17} & \alpha_{18} & \alpha_{55} & \alpha_{56} & \alpha_{57} & \alpha_{58} \\ \alpha_{25} & \alpha_{26} & \alpha_{27} & \alpha_{28} & \bar{\alpha}_{56} & \beta_{66} + \alpha_{66} & \alpha_{67} & \alpha_{68} \\ \alpha_{35} & \alpha_{36} & \alpha_{37} & \alpha_{38} & \bar{\alpha}_{57} & \bar{\alpha}_{67} & \alpha_{77} & \alpha_{78} \\ 0 & 0 & 0 & 0 & \bar{\alpha}_{58} & \bar{\alpha}_{68} & \bar{\alpha}_{78} & \alpha_{88} \end{pmatrix}. \quad (35)$$

<sup>6</sup>Notice that a straightforward application of the PPT criterion shows that these states are entangled.

In our discussion, we shall use the notation  $PM(j_1, \dots, j_l)$  to denote the principal minor calculated as the determinant of the sub-matrix obtained by selecting the  $(j_1, \dots, j_l)$  rows and columns. For example  $PM(1, 2)$  denotes the principal minor of order 2 obtained by calculating the determinant of the matrix at the intersection of rows and columns 1 and 2. We shall use the Sylvester criterion for a positive semi-definite matrix which says that an Hermitian matrix is positive semi-definite if and only if all principal minors are nonnegative (see, e.g., [1], [6]).

Assume that  $\rho$  is a PPT state. By applying Sylvester criterion with  $PM(4, 5)$ ,  $PM(4, 6)$ ,  $PM(4, 7)$  in (35), we obtain that we must have  $\alpha_{18} = \alpha_{28} = \alpha_{38} = 0$ . That is,  $p_8 = 0$  or  $p_1 = p_2 = p_3 = 0$ . However, if  $p_1 = p_2 = p_3 = 0$ ,  $PM(2, 5) = -\beta_{16}^2 < 0$ , which is not possible. This establishes that  $p_8 = 0$ .

With this assumption, consider  $PM(3, 5, 7)$  for (35). A direct calculation shows

$$PM(3, 5, 7) = \alpha_{77} (\overline{\alpha_{15}}\alpha_{37} + \alpha_{15}\overline{\alpha_{37}} - \alpha_{55}\alpha_{33} - \alpha_{11}\alpha_{77}) = -(1 - \lambda)^2 |p_3 p_5 - \overline{p_1 p_7}|^2.$$

The last expression is positive only if  $p_3 p_5 = \overline{p_1 p_7}$ . This implies

$$\alpha_{33}\alpha_{55} = \alpha_{11}\alpha_{77}. \quad (36)$$

We now show that (36) cannot be with  $\alpha_{77} \neq 0$ , therefore showing that  $p_7$  must be zero. Assume that (36) is true and  $\alpha_{11} = 0$ . Then at least one between  $\alpha_{55}$  and  $\alpha_{33}$  must be zero. However  $\alpha_{55}$  cannot be zero, because this would give  $PM(2, 5) = -\beta_{16}^2 < 0$  and  $\alpha_{33} = 0$  would require  $PM(3, 6) = -\alpha_{22}\alpha_{77} \geq 0$ , that is  $\alpha_{22} = 0$  which would lead again to  $PM(2, 5) = -\beta_{16}^2 < 0$ . Therefore, we must have  $\alpha_{11} \neq 0$ , which also, from orthogonality, implies  $\alpha_{66} \neq 0$  and from (36)  $\alpha_{33} \neq 0$  and  $\alpha_{55} \neq 0$ . Moreover  $\alpha_{22} \neq 0$  also is true by considering  $PM(2, 7)$  in (35). Therefore, we are in the situation where *all* the components of  $\psi_2$ , except  $p_4$  and  $p_8$ , are different from zero. Now, an argument as for  $PM(3, 5, 7)$  above, applied this time on  $PM(2, 3, 6)$ , along with the fact that  $\alpha_{22} \neq 0$ , gives

$$\alpha_{66}\alpha_{33} = \alpha_{22}\alpha_{77}, \quad (37)$$

and

$$\alpha_{23}\overline{\alpha_{67}} + \overline{\alpha_{23}}\alpha_{67} = \alpha_{22}\alpha_{77} + \alpha_{33}\alpha_{66} = 2\alpha_{33}\alpha_{66}. \quad (38)$$

Combining (36) with (37), we have

$$\alpha_{11}\alpha_{66} = \alpha_{22}\alpha_{55}. \quad (39)$$

We chose the overall phase of  $\psi^{(2)}$  such that  $q_1^2 p_1^2 = q_6^2 p_6^2$  is real. Hence,  $p_1 \overline{p_6} = \overline{p_1} p_6$ , i.e.  $\overline{\alpha_{16}} = \alpha_{16}$ . By multiplying (38) by  $\alpha_{16}$ , we obtain

$$\alpha_{23}\overline{\alpha_{17}} + \alpha_{17}\overline{\alpha_{23}} = 2\alpha_{16}\alpha_{33}. \quad (40)$$

Calculation of  $PM(2, 3, 5)$  gives, because of (36),

$$PM(2, 3, 5) = -\alpha_{22}\alpha_{33}\alpha_{55} + (\beta_{16} + \alpha_{16}) (\alpha_{23}\overline{\alpha_{17}} + \alpha_{17}\overline{\alpha_{23}}) - \alpha_{33} (\beta_{16} + \alpha_{16})^2.$$

By replacing (40) and using (39), this expression simplifies to

$$PM(2, 3, 5) = -\alpha_{33} (\alpha_{16} - (\beta_{16} + \alpha_{16}))^2 = -\alpha_{33}\beta_{16}^2 < 0.$$

This is not possible. Hence, (36) holds only if  $p_7 = 0$ .

Since  $p_4 = p_7 = p_8 = 0$ , consideration of  $PM(2, 7)$  and  $PM(1, 7)$  in (35) shows that it must be  $p_3 = 0$ , or  $p_6$  and  $p_5$  both equal to zero. However, the second case would imply  $PM(2, 5) = -\beta_{16}^2 < 0$ . This establishes  $p_3 = 0$  and concludes the proof of the necessity of  $p_3 = p_4 = p_7 = p_8 = 0$ . This shows that if a state is *PPT* its canonical form is written as (17).

In order for  $\rho$  to be a *PPT*-state the  $4 \times 4$  matrix  $\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^\dagger & \rho_{22} \end{pmatrix}$  must be *PPT* as a  $2 \times 2$  state, but since the *PPT* test is necessary and sufficient for separability in the  $2 \times 2$  case, this represents a  $2 \times 2$  separable

state. That is, there exist positive constants  $\mu_j$ ,  $j = 1, \dots, l$ , with  $\sum_{j=1}^l \mu_j = 1$  and  $2 \times 2$  density matrices  $\rho_j^{(1)}, \rho_j^{(2)}$  such that

$$\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^\dagger & \rho_{22} \end{pmatrix} = \sum_{j=1}^l \mu_j \rho_j^{(1)} \otimes \rho_j^{(2)}. \quad (41)$$

In particular,

$$\rho_{11} = \sum \mu_j \left( \rho_j^{(1)} \right)_{11} \rho_j^{(2)}, \quad \rho_{12} = \sum \mu_j \left( \rho_j^{(1)} \right)_{12} \rho_j^{(2)}, \quad \rho_{22} = \sum \mu_j \left( \rho_j^{(1)} \right)_{22} \rho_j^{(2)}. \quad (42)$$

The  $4 \times 4$  matrices

$$\tilde{\rho}_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \rho_j^{(2)},$$

are density matrices and, using (42), (41) and (17), we obtain

$$\rho = \sum \mu_j \rho_j^{(1)} \otimes \tilde{\rho}_j,$$

which shows that  $\rho$  is separable as well.

The fact that  $\rho$  in the form (17) is a PPT-state follows from the above characterization of  $\rho$  as separable and the fact that every separable state is a PPT-state.  $\square$